$\mathcal{N}{=}4$ superconformal Calogero models

Anton Galajinsky and Kirill Polovnikov

Laboratory of Mathematical Physics, Tomsk Polytechnic University,
Lenin Ave. 30, 634050 Tomsk, Russian Federation
E-mail: `galajin@mph.phtd.tpu.edu.ru`, `kir@mph.phtd.tpu.edu.ru`

Olaf Lechtenfeld

Institut für Theoretische Physik, Leibniz Universität Hannover,
Appelstrasse 2, D-30167 Hannover, Germany
E-mail: `lechtenf@itp.uni-hannover.de`

ABSTRACT: We continue the research initiated in hep-th/0607215 and apply our method of conformal automorphisms to generate various $\mathcal{N}{=}4$ superconformal quantum many-body systems on the real line from a set of decoupled particles extended by fermionic degrees of freedom. The $su(1,1|2)$ invariant models are governed by two scalar potentials obeying a system of nonlinear partial differential equations which generalizes the Witten-Dijkgraaf-Verlinde-Verlinde equations. As an application, the $\mathcal{N}{=}4$ superconformal extension of the three-particle (A-type) Calogero model generates a unique G_2-type Hamiltonian featuring three-body interactions. We fully analyze the $\mathcal{N}{=}4$ superconformal three- and four-particle models based on the root systems of $A_1 \oplus G_2$ and F_4, respectively. Beyond Wyllard's solutions we find a list of new models, whose translational non-invariance of the center-of-mass motion fails to decouple and extends even to the relative particle motion.

Contents

1. Introduction

Recently conformally invariant models in one dimension were investigated extensively [1–17]. On the one hand, the interest derives from the AdS/CFT correspondence. Although there has been considerable progress in understanding the AdS/CFT duality [18], nontrivial examples of AdS_2/CFT_1 correspondence are unknown. On the other hand, the conformal group $SO(2, d{-}1)$ is the isometry group of anti de Sitter space AdS_d. Since anti de Sitter space describes the near-horizon geometry of a wide class of extreme black holes (for a review see e.g. [19]), it was conjectured [20, 21] that the study of conformally invariant models in $d{=}1$ yields new insight into the quantum mechanics of black holes. This idea was pushed further in a series of papers [22–27], where some conformal mechanics on black-hole moduli spaces in $d{=}4$ and $d{=}5$ was constructed and investigated.

Particularly appealing in this context seems a proposal in [21] that an $\mathcal{N}{=}4$ superconformal extension of the Calogero model [28] might provide a microscopic description of the extreme Reissner-Nordström black hole near the horizon. It should be stressed, however, that the Calogero model, which describes a pair-wise interaction of n identical particles on the real line, is not the only multi-particle exactly solvable conformal mechanics available in $d{=}1$. More complicated systems describing three-particle and four-particle interactions were studied in [29–32]. Since in the context of [21] it is the structure of the conformal

algebra which matters, a priori any multi-particle $\mathcal{N}{=}4$ superconformal mechanics seems to be a good starting point. A classification of (off-shell) $d{=}1$ supermultiplets is interesting in its own right because of features absent in higher dimensions (see e.g. [33]). In this connection the construction of multi-particle $\mathcal{N}{=}4$ superconformal models is relevant for possible couplings of $d{=}1$, $\mathcal{N}{=}4$ supermultiplets.

Several attempts have been made to construct an $\mathcal{N}{=}4$ superconformal extension of the Calogero model [34–37]. In [34] conditions for $su(1,1|2)$ invariance were formulated, and some solutions were presented. In [35] the problem was solved for a complexification of the Calogero model. In [36, 37] the construction of an $\mathcal{N}{=}4$ superconformal Calogero model was reduced to solving a system of nonlinear partial differential equations, which generalizes the Witten-Dijkgraaf-Verlinde-Verlinde equation known from two-dimensional topological field theory [38, 39]. However, beyond the two-particle case only partial results were obtained.

In the present work we continue the research initiated in [40] and apply the method of unitary transformations to generate various $su(1,1|2)$ invariant quantum many-body systems, including an $\mathcal{N}{=}4$ superconformal extension of the Calogero model. In section 2 we discuss a specific unitary transformation, which maps a generic conformally invariant model of n identical particles on the real line to a set of decoupled particles, with the interaction being pushed into a nonlocal conformal boost generator. In this description, an $\mathcal{N}{=}4$ supersymmetric extension is straightforward to construct as we demonstrate in section 3. Both the conformal boost generator and its superpartner are nonlocal in this picture. The inverse transformation then provides us with the interacting Hamiltonian. The closure of the superconformal algebra poses constraints on the interaction, which are detailed and partially solved in section 4. Our superconformal models are governed by two scalar potentials obeying certain homogeneity conditions and the Witten-Dijkgraaf-Verlinde-Verlinde-type equations of [36, 37]. Explicit three- and four-particle solutions to these "structure equations" for the two scalar potentials are discussed in section 5 and found to be based on certain root systems. Beyond the models found by Wyllard [34], we present a list of solutions which break translation invariance not only for the center-of-mass motion but also for the relative motion. In section 6 we summarize our results and discuss possible further developments.

2. Conformal mechanics in a free nonlocal representation

Let us consider a system of n identical particles on the real line with a Hamiltonian of the generic form

$$H \;=\; \frac{1}{2m}p_i p_i \;+\; V_B(x^1,\ldots,x^n)\,, \tag{2.1}$$

where m stands for the mass of each particle. Throughout the paper a summation over repeated indices is understood. Later, the bosonic potential V_B will get supersymmetrically extended to a potential V including V_B.

For conformally invariant models the Hamiltonian H is part of the $so(1,2)$ conformal algebra

$$[D,H] \;=\; -\mathrm{i}\hbar H\,, \quad [H,K] \;=\; 2\mathrm{i}\hbar D\,, \quad [D,K] \;=\; \mathrm{i}\hbar K\,, \tag{2.2}$$

where D and K are the dilatation and conformal boost generators, respectively. Their realization in term of coordinates and momenta, subject to

$$[x^i, p_j] = \mathrm{i}\hbar\delta_j{}^i \,, \tag{2.3}$$

reads

$$D = -\frac{1}{4}(x^i p_i + p_i x^i) = D_0 \qquad \text{and} \qquad K = \frac{m}{2}x^i x^i = K_0 \,, \tag{2.4}$$

where the 0 subscript indicates the generators in the free model ($V_B{=}0$). The first relation in (2.2) restricts the potential via

$$(x^i\partial_i + 2)\,V_B = 0 \,, \tag{2.5}$$

meaning that V_B must be homogeneous of degree -2 for the model to be conformally invariant. In this paper we assume this to be the case. Two simple solutions to (2.5) are the free model of n non-interacting particles,

$$V_B = 0 \qquad \longrightarrow \qquad H_0 = \frac{1}{2m}p_i p_i \,, \tag{2.6}$$

and the Calogero model of n particles interacting through an inverse-square pair potential,

$$V_B = \sum_{i<j}\frac{g^2}{(x^i - x^j)^2} \qquad \longrightarrow \qquad H = H_0 + V_B \,. \tag{2.7}$$

As the next step we study the behavior of a generic conformal multi-particle mechanics under a judiciously chosen conformal-algebra automorphism. Given the particular $so(1,2)$ element

$$A = \alpha H - 2D + \frac{1}{\alpha}K \tag{2.8}$$

for a real parameter α, let us consider the unitary transformation

$$T \quad\longmapsto\quad T' = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}A}\,T\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}A} \tag{2.9}$$

on the $so(1,2)$ generators:[1]

$$H \longmapsto H' = \frac{1}{\alpha^2}K \,, \tag{2.10}$$

$$D \longmapsto D' = -D + \frac{2}{\alpha}K \,, \tag{2.11}$$

$$K \longmapsto K' = \alpha^2 H - 4\alpha D + 4K \,. \tag{2.12}$$

Notice that in the previous consideration it is only the structure of the conformal algebra that matters. Therefore, an analogous map exists for the free theory defined by (H_0, D_0, K_0):

$$T_0 \quad\longmapsto\quad T_0' = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}A_0}\,T_0\,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}A_0} \qquad \text{with} \quad A_0 = \alpha H_0 - 2D_0 + \frac{1}{\alpha}K_0 \,. \tag{2.13}$$

---

[1]A was chosen such that the Baker-Haussdorff series in (2.9) terminates at the third step [40].

This suggests the idea to combine the A-map with inverse A_0-map to link H and H_0 in the following scheme:

$$
\begin{array}{ccccc}
(H, D, K) & \xmapsto{\ A\ } & (H', D', K') & = & (H'_0, D'_0, K'_0 + \alpha^2 V_B) \\
(\widetilde{H}, \widetilde{D}, \widetilde{K}) & = & (H_0, D_0, K_0 + \alpha^2 \widehat{V}_B) & \xleftarrow{\ -A_0\ } & (H'_0, D'_0, K'_0 + \alpha^2 V_B)
\end{array}
\tag{2.14}
$$

with the abbreviation

$$
\widehat{V}_B \ = \ \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} A_0} \, V_B \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} A_0} \qquad \text{in} \qquad \widetilde{K} \ = \ K_0 + \alpha^2 \widehat{V}_B \, .
\tag{2.15}
$$

We remark that the dimensionful parameter α simply takes care of the different dimensionalities of the $so(1,2)$ generators and drops out of the final results as was shown in [40]. For the remainder of the paper we set $m = 1$.

Thus, with the help of a unitary operation one can transform a generic multi-particle conformal mechanics (2.1), (2.5) into a one describing a system of non-interacting particles. A peculiar feature of this correspondence is that the generator of special conformal transformations $\widetilde{K}$ is *nonlocal* and effectively hides the interaction potential. In fact, the interaction has disappeared in the Hamiltonian $\widetilde{H}$ but resurfaced in a nonlocal contribution to the conformal boost $\widetilde{K}$. Hence, the price paid for the simplification of the dynamics is a nonlocal realization of the full conformal algebra in the Hilbert space of the quantized conformal mechanics.

As an example, let us consider the conformal Calogero model describing the inverse-square pair-wise interaction of n identical particles of unit mass on the real line,

$$
V_B = \sum_{i<j}^{n} \frac{g^2}{\left(x^i - x^j\right)^2} \, ,
\tag{2.16}
$$

where g is the coupling constant. For this model, a map of H to H_0 similar to ours was constructed in [41]. However, the entire $so(1,2)$ algebra was not examined, and the nonlocal structure present in $\widetilde{K}$ was not revealed there. The quantum mechanical scattering analysis of the conformal Calogero model was accomplished in [42], where it was argued that the particles merely exchange their asymptotic momenta without altering their values. The asymptotic wave function only picks up an energy-independent phase factor through the scattering process. Since the conformal Calogero particles are indistinguishable, their physics is that of n free bosons. Thus, the general consideration presented above is in agreement with [42, 43].

3. $\mathcal{N}=4$ superconformal extension

The unitary map of a generic multi-particle conformal mechanics to a set of decoupled particles considered in the previous section offers a novel way to constructing superconformal extensions. In our setting this amounts to properly adding fermionic degrees of freedom to a free system and modifying the nonlocal boost generator $\widetilde{K}$ so as to close an N-extended superconformal algebra. Application of the inverse unitary transformation to the set of free

superparticles then produces a desired superconformal extension of the original interacting conformal mechanics. In this section we discuss the corresponding algebraic framework.

The bosonic sector of the $\mathcal{N}{=}4$ superconformal algebra $su(1,1|2)$ includes two subalgebras. Along with $so(1,2)$ considered in the previous section one also finds the $su(2)$ R-symmetry subalgebra generated by J_a with $a = 1, 2, 3$. The fermionic sector is exhausted by the $su(2)$ doublet supersymmetry generators Q_α and $\bar{Q}^\alpha$ as well as their superconformal partners S_α and $\bar{S}^\alpha$, with $\alpha = 1, 2$, subject to the hermiticity relations

$$(Q_\alpha)^\dagger = \bar{Q}^\alpha \qquad \text{and} \qquad (S_\alpha)^\dagger = \bar{S}^\alpha . \tag{3.1}$$

The bosonic generators are hermitian. The non-vanishing (anti)commutation relations in our superconformal algebra read[2]

$$
\begin{aligned}
[D, H] &= -\mathrm{i}\hbar\, H\,, & [H, K] &= 2\mathrm{i}\hbar\, D\,, \\
[D, K] &= +\mathrm{i}\hbar\, K\,, & [J_a, J_b] &= \mathrm{i}\hbar\, \epsilon_{abc} J_c\,, \\
\{Q_\alpha, \bar{Q}^\beta\} &= 2\hbar\, H \delta_\alpha{}^\beta\,, & \{Q_\alpha, \bar{S}^\beta\} &= +2\mathrm{i}\hbar\,(\sigma_a)_\alpha{}^\beta J_a - 2\hbar\, D\delta_\alpha{}^\beta - \mathrm{i}\hbar\, C\delta_\alpha{}^\beta\,, \\
\{S_\alpha, \bar{S}^\beta\} &= 2\hbar\, K \delta_\alpha{}^\beta\,, & \{\bar{Q}^\alpha, S_\beta\} &= -2\mathrm{i}\hbar\,(\sigma_a)_\beta{}^\alpha J_a - 2\hbar\, D\delta_\beta{}^\alpha + \mathrm{i}\hbar\, C\delta_\beta{}^\alpha\,, \\
[D, Q_\alpha] &= -\tfrac{1}{2}\mathrm{i}\hbar\, Q_\alpha\,, & [D, S_\alpha] &= +\tfrac{1}{2}\mathrm{i}\hbar\, S_\alpha\,, \\
[K, Q_\alpha] &= +\mathrm{i}\hbar\, S_\alpha\,, & [H, S_\alpha] &= -\mathrm{i}\hbar\, Q_\alpha\,, \\
[J_a, Q_\alpha] &= -\tfrac{1}{2}\hbar\,(\sigma_a)_\alpha{}^\beta Q_\beta\,, & [J_a, S_\alpha] &= -\tfrac{1}{2}\hbar\,(\sigma_a)_\alpha{}^\beta S_\beta\,, \\
[D, \bar{Q}^\alpha] &= -\tfrac{1}{2}\mathrm{i}\hbar\, \bar{Q}^\alpha\,, & [D, \bar{S}^\alpha] &= +\tfrac{1}{2}\mathrm{i}\hbar\, \bar{S}^\alpha\,, \\
[K, \bar{Q}^\alpha] &= +\mathrm{i}\hbar\, \bar{S}^\alpha\,, & [H, \bar{S}^\alpha] &= -\mathrm{i}\hbar\, \bar{Q}^\alpha\,, \\
[J_a, \bar{Q}^\alpha] &= \tfrac{1}{2}\hbar\, \bar{Q}^\beta (\sigma_a)_\beta{}^\alpha\,, & [J_a, \bar{S}^\alpha] &= \tfrac{1}{2}\hbar\, \bar{S}^\beta (\sigma_a)_\beta{}^\alpha\,.
\end{aligned}
\tag{3.2}
$$

Here $\epsilon_{123} = 1$, and C stands for the central charge.

Following the same strategy as in the previous section, we employ the conformal automorphism (2.9) and its free inverse as indicated in (2.14), with A being of the same form as in (2.8). It is very plausible that the new (tilded) generators differ from the free ones only in the instances of K, S_α and $\bar{S}^\alpha$, so we write (omitting the complex conjugates and suppressing the indices)

$$
\begin{aligned}
H &\longmapsto & \widetilde{H} &= H_0\,, \\
D &\longmapsto & \widetilde{D} &= D_0\,, \\
K &\longmapsto & \widetilde{K} &= K_0 + \alpha^2 \widehat{V}\,, \\
Q &\longmapsto & \widetilde{Q} &= Q_0\,, \\
S &\longmapsto & \widetilde{S} &= S_0 - \alpha\frac{\mathrm{i}}{\hbar}\widehat{[S_0, V]}\,, \\
J &\longmapsto & \widetilde{J} &= J_0\,,
\end{aligned}
\tag{3.3}
$$

---

[2]σ_1, σ_2 and σ_3 denote the Pauli matrices.

where the correction to S_0 is determined from the form of $\widetilde{K}$ through the $[\widetilde{K}, \widetilde{Q}]$ commutator in (3.2), and we again use the notation

$$\widehat{T} \;=\; \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} A_0} \, T \, \mathrm{e}^{\frac{\mathrm{i}}{\hbar} A_0} \;. \tag{3.4}$$

Note that we have written V instead of V_B, anticipating fermionic and quantum contributions to the Hamiltonian

$$H \;=\; H_0 \,+\, V \qquad \text{with} \qquad V \;=\; V_B + V_F + O(\hbar) \;. \tag{3.5}$$

Given V, the $[H, S]$ and $[H, \bar{S}]$ commutators in (3.2) enforce an interacting part for the supersymmetry generators,

$$Q_\alpha \;=\; Q_{0\alpha} - \frac{\mathrm{i}}{\hbar} \, [S_{0\alpha}, V] \qquad \text{and} \qquad \bar{Q}^\alpha \;=\; \bar{Q}_0^\alpha - \frac{\mathrm{i}}{\hbar} \, [\bar{S}_0^\alpha, V] \,, \tag{3.6}$$

while all other generators T remain free, i.e.

$$D \;=\; D_0 \,, \qquad K \;=\; K_0 \,, \qquad S \;=\; S_0 \qquad \text{and} \qquad J \;=\; J_0 \;. \tag{3.7}$$

This is the result of inverting the map (3.3) to return from the tilded generators $\widetilde{T}$ to the original ones T. We shall, however, use the tilded generators (3.3) to find the form of V.

For a mechanical realization of the $su(1,1|2)$ superalgebra, one introduces fermionic degrees of freedom represented by the operators ψ_α^i and $\bar{\psi}^{i\alpha}$, with $i = 1, \ldots, n$ and $\alpha = 1, 2$, which are hermitian conjugates of each other and obey the anti-commutation relations[3]

$$\{\psi_\alpha^i, \psi_\beta^j\} \;=\; 0 \,, \qquad \{\bar{\psi}^{i\alpha}, \bar{\psi}^{j\beta}\} \;=\; 0 \,, \qquad \{\psi_\alpha^i, \bar{\psi}^{j\beta}\} \;=\; \hbar \, \delta_\alpha{}^\beta \delta^{ij} \;. \tag{3.8}$$

In the extended space it is easy to construct the free fermionic generators associated with the free Hamiltonian $H_0 = \frac{1}{2} p_i p_i$, namely (for $m{=}1$)

$$Q_{0\alpha} \;=\; p_i \psi_\alpha^i \,, \qquad \bar{Q}_0^\alpha \;=\; p_i \bar{\psi}^{i\alpha} \qquad \text{and} \qquad S_{0\alpha} \;=\; x^i \psi_\alpha^i \,, \qquad \bar{S}_0^\alpha \;=\; x^i \bar{\psi}^{i\alpha} \,, \tag{3.9}$$

as well as $su(2)$ generators

$$J_{0a} \;=\; \frac{1}{2} \bar{\psi}^{i\alpha} (\sigma_a)_\alpha{}^\beta \psi_\beta^i \;. \tag{3.10}$$

Notice that these are automatically Weyl-ordered. The free dilatation and conformal boost operators maintain their bosonic form

$$D_0 \;=\; -\frac{1}{4}(x^i p_i + p_i x^i) \qquad \text{and} \qquad K_0 \;=\; \frac{1}{2} x^i x^i \;. \tag{3.11}$$

In contrast to the bosonic case, the free generators T_0 fail to satisfy the full algebra (3.2). Even for $C{=}0$, the $\{Q, \bar{S}\}$ and $\{\bar{Q}, S\}$ anticommutators require corrections cubic in the fermions, which we can restrict to Q and $\bar{Q}$ as in (3.6). Dimensional analysis reveals that the coefficients of these cubic terms have a dimension of length^{-1} and thus cannot be constants. It follows further that H contains quadratic and quartic fermionic terms, which

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[3]Spinor indices are raised and lowered with the invariant tensor $\epsilon^{\alpha\beta}$ and its inverse $\epsilon_{\alpha\beta}$, where $\epsilon^{12} = 1$.

are collected in V_F in (3.5). Hence, even for $V_B{=}0$ there does not exist a free mechanical representation of the algebra (3.2).

The generators $\widetilde{K}$, $\widetilde{S}$ and $\widetilde{\bar{S}}$ are nonlocal. Substituting their form (3.3) into the super-conformal algebra (3.2) one gets a set of restrictions on the form of the operator V:

$$
\begin{aligned}
&[K_0, V] = 0, \qquad [D_0, V] = -\mathrm{i}\hbar V, \qquad\qquad [J_{0a}, V] = 0, \\
&\{S_{0\alpha}, [S_{0\beta}, V]\} = \hbar^2 \psi^i_\alpha \psi^i_\beta, \qquad\qquad\qquad\qquad \{\bar{S}^\alpha_0, [\bar{S}^\beta_0, V]\} = \hbar^2 \bar\psi^{i\alpha}\bar\psi^{i\beta}, \\
&\{S_{0\alpha}, [\bar{S}^\beta_0, V]\} = +2\hbar^2 (\sigma_a)_\alpha{}^\beta J_{0a} + \frac{1}{2}\hbar^2(\psi^i_\alpha\bar\psi^{i\beta} - \bar\psi^{i\beta}\psi^i_\alpha) - \hbar^2 C\delta_\alpha{}^\beta, \\
&\{\bar{S}^\alpha_0, [S_{0\beta}, V]\} = -2\hbar^2 (\sigma_a)_\beta{}^\alpha J_{0a} - \frac{1}{2}\hbar^2(\psi^i_\beta\bar\psi^{i\alpha} - \bar\psi^{i\alpha}\psi^i_\beta) + \hbar^2 C\delta_\beta{}^\alpha, \\
&\{[S_{0\alpha}, V], [S_{0\beta}, V]\} + \mathrm{i}\hbar\{Q_{0\alpha}, [S_{0\beta}, V]\} + \mathrm{i}\hbar\{Q_{0\beta}, [S_{0\alpha}, V]\} = 0, \\
&\{[\bar{S}^\alpha_0, V], [\bar{S}^\beta_0, V]\} + \mathrm{i}\hbar\{\bar{Q}^\alpha_0, [\bar{S}^\beta_0, V]\} + \mathrm{i}\hbar\{\bar{Q}^\beta_0, [\bar{S}^\alpha_0, V]\} = 0, \\
&\{[S_{0\alpha}, V], [\bar{S}^\beta_0, V]\} + \mathrm{i}\hbar\{Q_{0\alpha}, [\bar{S}^\beta_0, V]\} + \mathrm{i}\hbar\{\bar{Q}^\beta_0, [S_{0\alpha}, V]\} + 2\hbar^3 V\delta_\alpha{}^\beta = 0, \\
&\mathrm{i}\hbar[Q_{0\alpha}, V] - [H_0 + V, [S_{0\alpha}, V]] = 0, \qquad\qquad \mathrm{i}\hbar[\bar{Q}^\alpha_0, V] - [H_0 + V, [\bar{S}^\alpha_0, V]] = 0.
\end{aligned}
$$

(3.12)

Notice that the vanishing (anti)commutators discarded in (3.2) should be taken into account as they also give constraints on V. For obtaining (3.12) the following identities are helpful:

$$
\begin{aligned}
\widehat{Q}_{0\alpha} &= 2Q_{0\alpha} + \frac{1}{\alpha}S_{0\alpha}, & \widehat{\bar{Q}}^\alpha_0 &= 2\bar{Q}^\alpha_0 + \frac{1}{\alpha}\bar{S}^\alpha_0, \\
\widehat{S}_{0\alpha} &= -\alpha Q_{0\alpha}, & \widehat{\bar{S}}^\alpha_0 &= -\alpha\bar{Q}^\alpha_0, \\
S_{0\alpha} &= \alpha\widehat{Q}_{0\alpha} + 2\widehat{S}_{0\alpha}, & \bar{S}^\alpha_0 &= \alpha\widehat{\bar{Q}}^\alpha_0 + 2\widehat{\bar{S}}^\alpha_0.
\end{aligned}
$$

(3.13)

4. The structure equations

Let us discuss the structure of solutions to the constraints (3.12). The first line in (3.12) implies that the potential $V = V_B + V_F + O(\hbar)$ transforms as a scalar under $su(2)$ and is a degree -2 homogeneous function of the x^i. It is straightforward to check that an ansatz for V_F quadratic in ψ^i and $\bar\psi^i$ fails to solve (3.12). This is in contrast with $\mathcal{N}{=}2$ superconformal extensions [40, 44]. Thus it seems natural to try a general ansatz quartic in the fermionic coordinates,[4]

$$
V = V_B(x) + \hbar O_1(x) + \hbar^2 O_2(x) + M_{ij}(x)\langle\psi^i_\alpha\bar\psi^{j\alpha}\rangle + \frac{1}{4}L_{ijkl}(x)\langle\psi^i_\alpha\psi^{j\alpha}\bar\psi^{k\beta}\bar\psi^l_\beta\rangle, \quad (4.1)
$$

with completely symmetric unknown functions M_{ij} and L_{ijkl}. Here, the symbol $\langle\ldots\rangle$ stands for symmetric (or Weyl) ordering (for our conventions see appendix A), and the contributions $\hbar O_1(x)$ and $\hbar^2 O_2(x)$ were included to account for the ordering ambiguity present in the fermionic sector. The argument x indicates dependence on $\{x^1, \ldots, x^n\}$.

Introducing the notations

$$
L_{ijkl}\, x^l =: -W_{ijk} \qquad \text{and} \qquad M_{ij}\, x^j =: Y_i \qquad (4.2)
$$

---

[4]The classical consideration in [37] implies that (4.1) is indeed the most general quartic ansatz compatible with the $\mathcal{N}{=}4$ superconformal algebra.

and substituting the ansatz (4.1) into the constraints (3.12), one obtains the following system of partial differential and algebraic "structure equations",

$$L_{ijkl} = \partial_i W_{jkl} = \partial_j W_{ikl}\,, \qquad\qquad M_{ij} = -\partial_i Y_j = -\partial_j Y_i\,, \qquad (4.3)$$

$$x^i W_{ijk} = -\delta_{jk}\,, \qquad\qquad x^i Y_i = -C\,, \qquad (4.4)$$

$$M_{ij} + W_{ijk}Y_k = 0\,, \qquad\qquad W_{ikp}W_{jlp} = W_{jkp}W_{ilp}\,, \qquad (4.5)$$

as well as a boundary condition on Y_i,

$$\frac{1}{2}Y_i Y_i \;=\; V_B\,. \qquad (4.6)$$

Besides, one determines the quantum corrections as

$$O_1 \;=\; 0 \qquad\text{and}\qquad O_2 \;=\; \frac{1}{8}W_{ijk}W_{ijk}\,. \qquad (4.7)$$

In contrast to $\mathcal{N}=2$ superconformal models, here the algebra requires a nontrivial quantum correction. The explicit derivation of (4.3)–(4.7) is tedious and most efficiently achieved using reordering relations given in appendix A.

Taking into account that W_{ijk} is a completely symmetric function, from (4.3) one finds

$$\begin{aligned}
W_{ijk} &= \partial_i\partial_j\partial_k F &\Leftrightarrow&\qquad L_{ijkl} = \partial_i\partial_j\partial_k\partial_l F\,,\\
Y_i &= \partial_i U &\Leftrightarrow&\qquad M_{ij} = -\partial_i\partial_j U\,,
\end{aligned} \qquad (4.8)$$

with two scalar potentials $F(x)$ and $U(x)$ to be determined. Thus, these scalars govern the $\mathcal{N}=4$ superconformal extension and obey the following system of nonlinear partial differential equations,

$$(\partial_i\partial_k\partial_p F)(\partial_j\partial_l\partial_p F) = (\partial_j\partial_k\partial_p F)(\partial_i\partial_l\partial_p F)\,, \qquad x^i\partial_i\partial_j\partial_k F = -\delta_{jk}\,, \quad (4.9)$$

$$\partial_i\partial_j U - (\partial_i\partial_j\partial_k F)\,\partial_k U = 0\,, \qquad \frac{1}{2}(\partial_i U)(\partial_i U) \;=\; V_B\,, \qquad x^i\partial_i U = -C\,. \quad (4.10)$$

Notice that F is defined modulo a quadratic polynomial while U is defined up to a constant.

Wyllard [34] obtained equivalent equations, but employed a different fermionic ordering. In contrast to his equations, $\hbar$ does not appear in (4.9) or (4.10), since our Weyl-ordering prescription matches smoothly to the classical limit. For the classical Calogero model similar equations were discussed in [37].

The right-most equations in (4.9) and (4.10) are inhomogeneous with constants δ_{jk} and C (the central charge) on the right-hand side and display an explicit coordinate dependence. Furthermore, the second equation in (4.9) can be integrated twice to obtain

$$x^i\partial_i F - 2F + \frac{1}{2}x^i x^i \;=\; 0\,, \qquad (4.11)$$

where we used the freedom in the definition of F to put the integration constants — a linear function on the right-hand side — to zero. It is important to realize that the inhomogeneous term in this integrated equation does break translation invariance and excludes the trivial solution $F = 0$ equivalent to a homogeneous quadratic polynomial. This effect is absent

in $\mathcal{N}=2$ superconformal models, where the four-fermion potential term is not needed and, hence, F does not appear [40]. This issue is also discussed in [34].

To be more explicit, we extract the center-of-mass dynamics by splitting

$$F \ = \ F_{\mathrm{com}}(X) + F_{\mathrm{rel}}(x) \qquad \mathrm{and} \qquad U \ = \ U_{\mathrm{com}}(X) + U_{\mathrm{rel}}(x) \tag{4.12}$$

with the center-of-mass coordinate $X := \frac{1}{n} \sum_{i=1}^{n} x^i$. If the *relative* particle motion is translation invariant (which need not be the case), then

$$\sum_{i=1}^{n} \partial_i F_{\mathrm{rel}} \ = \ 0 \ = \ \sum_{i=1}^{n} \partial_i U_{\mathrm{rel}} \tag{4.13}$$

and, applying $\sum_i \partial_i$ to (4.11) and the last equation in (4.10), we readily find

$$X\,F''_{\mathrm{com}} - F'_{\mathrm{com}} \ = \ -n\,X \qquad \mathrm{but} \qquad X\,U''_{\mathrm{com}} + U'_{\mathrm{com}} \ = \ 0\,, \tag{4.14}$$

which are solved by

$$F_{\mathrm{com}} \ = \ -\frac{n}{2}X^2 \ln|nX| + \lambda X^2 + \mu \qquad \mathrm{and} \qquad U_{\mathrm{com}} \ = \ -g_0 \ln|nX| + \nu \tag{4.15}$$

with free constants λ, μ, ν and g_0. Clearly, in this case we may put to zero U_{com} but not F_{com}, so that for $g_0{=}0$ we end up with a center-of-mass contribution

$$V_{\mathrm{com}} \ = \ \frac{\hbar^2}{8n}X^{-2} \ + \ \frac{n}{4}\,X^{-2}\,\langle \Psi_\alpha \Psi^\alpha \bar{\Psi}^\beta \bar{\Psi}_\beta \rangle \qquad \mathrm{with} \qquad \Psi_\alpha \ = \ \frac{1}{n}\sum_{i=1}^{n} \psi_\alpha^i\,. \tag{4.16}$$

Hence, one can separate a translation-invariant relative motion from the center-of-mass motion, but the latter is non-linear due to an X^{-2} potential as enforced by the superconformal algebra (3.2).

Our attack on (4.9) and (4.10) begins with the homogeneity conditions

$$(x^i\partial_i - 2)F \ = \ -\frac{1}{2}\,x^i x^i \qquad \mathrm{and} \qquad x^i\partial_i U \ = \ -C\,. \tag{4.17}$$

The most general solution is the sum of a particular solution and the general solution to the homogeneous equations,

$$(x^i\partial_i - 2)F_{\mathrm{hom}} \ = \ 0 \qquad \mathrm{and} \qquad x^i\partial_i U_{\mathrm{hom}} \ = \ 0\,, \tag{4.18}$$

which is spanned by the homogeneous functions of degree two and zero, respectively. For a particular solution to (4.17), we make the ansatz

$$F \ = \ -\sum_{\mu=0}^{d} h_\mu \frac{1}{2}(z^\mu)^2 \ln|z^\mu| \qquad \mathrm{and} \qquad U \ = \ -\sum_{\mu=0}^{d} g_\mu \ln|z^\mu| \tag{4.19}$$

with a certain number $(d{+}1)$ of linear coordinate combinations

$$z^\mu \ = \ n_i^\mu\,x^i \qquad \mathrm{beginning\ with} \qquad z^0 \ = \ nX \ = \ \sum_i x^i\,. \tag{4.20}$$

The relative motion is translation invariant if $\sum_i n_i^\mu = 0$ for $\mu > 0$. Compatibility with the conditions (4.17) directly yields

$$\sum_{\mu=0}^{d} h_\mu \, n_i^\mu n_j^\mu \;=\; \delta_{ij} \qquad \text{and} \qquad \sum_{\mu=0}^{d} g_\mu \;=\; C \;. \tag{4.21}$$

The second relation fixes the central charge, and the first relation amounts to a decomposition of the identity (δ_{ij}) into rank-one projectors. It turns out that the g_μ are independent free couplings (if not forced to zero) while the h_μ are not.

A *minimal* solution involves $d+1 = n$ mutually orthogonal vectors n^μ beginning with $\vec{n}^0 = (1, 1, \ldots, 1)$ and normalized as

$$\vec{n}^\mu \cdot \vec{n}^\nu \;\equiv\; \sum_i n_i^\mu n_i^\nu \;=\; h_\mu^{-1} \delta^{\mu\nu} \;. \tag{4.22}$$

From (4.19) we derive

$$W_{ijk} \;=\; -\sum_{\mu=0}^{n-1} h_\mu \, \frac{n_i^\mu n_j^\mu n_k^\mu}{z^\mu} \qquad \text{and} \qquad Y_i \;=\; -\sum_{\mu=0}^{n-1} g_\mu \, \frac{n_i^\mu}{z^\mu} \;, \tag{4.23}$$

and for the minimal choice (4.22) the bosonic potential becomes

$$V_B \;=\; \frac{1}{2} \sum_{\mu=0}^{n-1} \frac{g_\mu^2 h_\mu^{-1}}{(z^\mu)^2} \qquad \text{and} \qquad O_2 \;=\; \frac{1}{8} \sum_{\mu=0}^{n-1} \frac{h_\mu^{-1}}{(z^\mu)^2} \;, \tag{4.24}$$

which demonstrates that the quantum corrections only renormalize the coupling constants,

$$g_\mu^2 \qquad \longmapsto \qquad \widetilde{g}_\mu^2 \;=\; g_\mu^2 + \frac{1}{4}\hbar^2 \qquad \forall \, \mu \;. \tag{4.25}$$

It is instructive to first investigate small values of n. At $n=2$, relative translation invariance demands $\vec{n}^0 = (1,1)$ and $\vec{n}^1 = (1,-1)$ with $h_0 = h_1 = \frac{1}{2}$, whence

$$\begin{aligned}
F_{\text{rel}} &= -\frac{1}{4}\,(x^1 - x^2)^2 \ln|x^1 - x^2| \qquad & \text{and} \qquad & U_{\text{rel}} = -g_1 \ln|x^1 - x^2| \;, \\
W_{\ldots} &= -\frac{1}{2}\left(\frac{1}{x^1 + x^2} \pm \frac{1}{x^1 - x^2}\right) \qquad & \text{and} \qquad & V_B + \hbar^2 O_2 = \frac{\widetilde{g}_0^2}{(x^1 + x^2)^2} + \frac{\widetilde{g}_1^2}{(x^1 - x^2)^2} \;.
\end{aligned} \tag{4.26}$$

Beyond $n=2$, minimal choices are no longer invariant modulo sign under all permutations of the positions x^i, but, due to the linearity of (4.17), this can be remedied by finally summing over all permutations. The result is, in general, an overcomplete set of $d+1 > n$ non-orthogonal vectors. In section 5 we shall find a non-minimal one-parameter set (in F) of $n=3$ solutions to all structure equations for the choice

$$\vec{n}^0 = (1,1,1), \quad \vec{n}^1 = (1,-1,0), \quad \vec{n}^2 = (1,1,-2) \qquad \text{plus three permutations} \;. \tag{4.27}$$

However, a nontrivial U_{rel} based either on $\vec{n}^1$ or on $\vec{n}^2$ appears only for two specific parameter values. One may recognize here the root system of $A_1 \oplus G_2$, which is the even part of the root system of the Lie superalgebra G_3. In the same section, we will describe

five one-parameter families of $n=4$ solutions based on (parts of) the F_4 root system. Here, only three discrete models have U_{rel} non-vanishing, but for two of these the relative particle motion is not translation invariant.

In order to discover these and other solutions to the structure equations, within our ansatz (4.19)–(4.21) it remains to solve the two left-most equations in (4.9) and (4.10),

$$\partial_i Y_j - W_{ijk} Y_k \;=\; 0 \qquad \text{and} \qquad W_{ikp} W_{jlp} \;=\; W_{jkp} W_{ilp} \tag{4.28}$$

for $Y_i = \partial_i U$ and $W_{ijk} = \partial_i \partial_j \partial_k F$. This is quite tough because of their nonlinearity, and we address them in the following section. Already we notice, however, that the full system of structure equations (4.9) and (4.10) can be attacked in two different ways. One possibility, pursued in subsection 5.1, is to start with a given conformal potential V_B, e.g. of Calogero form, find a corresponding U, hence Y, and then search for a solution W to (4.28) before integrating it to F. Alternatively, as in subsection 5.2, one can take a particular solution F of the quadratic relations in (4.28), then find a solution Y to the first equation in (4.28) and integrate it to U, thereby determining V_B afterwards. The second strategy will yield $\mathcal{N}=4$ superconformal models generalizing the Calogero one. Finally, any full solution (Y, W) also determines the $su(1,1|2)$ generators as

$$\begin{aligned}
Q_\alpha &= (p_k + \mathrm{i} Y_k)\, \psi_\alpha^k \;+ \tfrac{\mathrm{i}}{2} W_{ijk} \,\langle \psi_\beta^i\, \psi^{j\beta} \bar\psi_\alpha^k \rangle\,, \\
\bar Q^\alpha &= (p_k - \mathrm{i} Y_k)\, \bar\psi^{k\alpha} + \tfrac{\mathrm{i}}{2} W_{ijk} \,\langle \psi^{i\alpha} \bar\psi^{j\beta} \bar\psi_\beta^k \rangle\,, \\
H &= \tfrac{1}{2} p_i p_i \;+ \tfrac{1}{2} Y_i Y_i \;+ \tfrac{\hbar^2}{8} W_{ijk} W_{ijk} \;-\; \partial_i Y_j\, \langle \psi_\alpha^i \bar\psi^{j\alpha} \rangle \;+\; \tfrac{1}{4} \partial_i W_{jkl}\, \langle \psi_\alpha^i \psi^{j\alpha} \bar\psi^{k\beta} \bar\psi_\beta^l \rangle\,,
\end{aligned} \tag{4.29}$$

while the other generators are of bilinear form given in (3.9), (3.10) and (3.11).

We conclude the section by observing a resemblance of the quadratic relations in (4.28) or (4.9) to an n-parametric potential deformation of an n-dimensional Fröbenius algebra [45], which plays an important role in two-dimensional topological field theory [38, 39]. Let us recall that an n-dimensional commutative associative algebra A with unit element e is called a Fröbenius algebra if it is supplied with a non-degenerate symmetric bilinear form obeying (for a review see e.g. [45])

$$\langle a \cdot b,\, c \rangle \;=\; \langle a,\, b \cdot c \rangle \qquad \forall\, a, b, c \in A \,. \tag{4.30}$$

Choosing a basis $\{e_i \,|\, i = 1, \dots, n\}$ with $e_1 = e$, one has

$$\langle e_i, e_j \rangle \;=\; \eta_{ij} \qquad \text{and} \qquad e_i \cdot e_j \;=\; f_{ij}{}^k\, e_k\,, \tag{4.31}$$

where η_{ij} is the metric with inverse η^{ij} and $f_{ij}{}^k$ are the structure constants. The commutativity and associativity of the algebra along with (4.30) produce the constraints

$$f_{ij}{}^k \;=\; f_{ji}{}^k\,, \qquad f_{1i}{}^j \;=\; \delta_i{}^j\,, \qquad f_{ij}{}^k \eta_{kl} \;=\; f_{lj}{}^k \eta_{ki}\,, \qquad f_{ij}{}^k f_{kl}{}^m \;=\; f_{lj}{}^k f_{ki}{}^m\,. \tag{4.32}$$

Thus, $f_{ij}{}^k \eta_{kl} = f_{ijl}$ is totally symmetric and subject to the quadratic relations above.

An n-parametric potential deformation of such a Fröbenius algebra is defined by a set of functions

$$f_{ijk}(x) = \partial_i \partial_j \partial_k F(x) \qquad (4.33)$$

descending from some scalar potential $F(x)$ with $x = \{x^1, \ldots, x^n\}$. To qualify as a deformation, these functions must satisfy the relations

$$f_{1ij}(x) = \eta_{ij}, \qquad \partial_i \eta_{jk} = 0, \qquad \eta^{kn} f_{ijk}(x) f_{lmn}(x) = \eta^{kn} f_{ljk}(x) f_{imn}(x), \quad (4.34)$$

which represent nonlinear partial differential equations for $F(x)$. In the context of two-dimensional topological field theory, F is known as the free energy, and (4.34) is called the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation [38, 39]. An interesting link between the WDVV equation and differential geometry was established in [45]. Comparing (4.9) with (4.34), we see that our algebra does not have a distinguished element serving as a unit element. Instead, the metric arises from the second equation in (4.9) by a contraction of f_{ijk} with the coordinates x^i.

5. Solutions to the structure equations

Proving the integrability of the structure equations (4.9) and (4.10) is a difficult task. For the WDVV equations this was done rigorously only for the simpler case of a decomposable Fröbenius algebra [45]. So, instead of trying to find a formal proof, we shall consider a few explicit examples and outline a simple constructive procedure how to integrate the structure equations. Finally, we give all solutions of the three- and four-particle cases which fit in our ansatz (4.19) with $A_1 \oplus G_2$ and F_4 positive root vectors, respectively.

5.1 Three-body $\mathcal{N}{=}4$ superconformal Calogero model

In this subsection we construct a particular solution to (4.9) and (4.10) or, equivalently, (4.3)–(4.6), for the case of three-body Calogero model governed by the potential

$$V_B = \sum_{i<j}^{3} \frac{g^2}{(x^i - x^j)^2}, \qquad (5.1)$$

leading to $C{=}3g$ because we sum over three permutations. It is easy to construct a corresponding U satisfying the second and third equation in (4.10). The general solution reads

$$U = -g \sum_{i<j}^{3} \ln |x^i - x^j| + L(y, z), \qquad (5.2)$$

where L is an arbitrary function of the ratios

$$y = \frac{x^1}{x^2} \qquad \text{and} \qquad z = \frac{x^1}{x^3} \qquad (5.3)$$

subject to

$$(\partial_i L)(\partial_i L) = g \sum_{i \neq j} \frac{\partial_i L - \partial_j L}{x^i - x^j}, \qquad (5.4)$$

and so we may put $L \equiv 0$, which we do for simplicity. Models based on the potential $U = -g \sum_{i<j} \ln |x^i - x^j|$ we term 'Calogero'.

Next we turn to the WDVV coefficients W_{ijk}, of which there are ten for $n=3$. The six linear relations in the first equation of (4.4) allow us to express the WDVV coefficients in terms of four objects. In order to find their explicit form, we integrate (4.11) to

$$F = -\frac{1}{2} \left(x^i x^i \ln |x^1| - (x^1)^2 \Delta(y, z) \right), \tag{5.5}$$

where $\Delta(y, z)$ is an unknown function to be determined below, and we have distinguished the x^1 coordinate. Triple differentiation of F yields

$$x^1 W_{111} = -1 - \frac{1}{y^2} - \frac{1}{z^2} + \Sigma_1 + \Sigma_2 + 3\Sigma_3 + 3\Sigma_4, \quad x^1 W_{123} = yz(\Sigma_3 + \Sigma_4),$$

$$x^1 W_{112} = \frac{1}{y} - y\Sigma_1 - y\Sigma_3 - 2y\Sigma_4, \qquad x^1 W_{113} = \frac{1}{z} - z\Sigma_2 - 2z\Sigma_3 - z\Sigma_4,$$

$$x^1 W_{122} = -1 + y^2 \Sigma_1 + y^2 \Sigma_4, \qquad x^1 W_{133} = -1 + z^2 \Sigma_2 + z^2 \Sigma_3,$$

$$x^1 W_{222} = -y^3 \Sigma_1, \qquad x^1 W_{223} = -zy^2 \Sigma_4, \qquad x^1 W_{233} = -yz^2 \Sigma_3, \qquad x^1 W_{333} = -z^3 \Sigma_2, \tag{5.6}$$

with four subsidiary functions

$$\Sigma_1 = \frac{1}{2} y^3 \frac{\partial^3 \Delta}{\partial y^3} + 3y^2 \frac{\partial^2 \Delta}{\partial y^2} + 3y \frac{\partial \Delta}{\partial y}, \qquad \Sigma_2 = \frac{1}{2} z^3 \frac{\partial^3 \Delta}{\partial z^3} + 3z^2 \frac{\partial^2 \Delta}{\partial z^2} + 3z \frac{\partial \Delta}{\partial z},$$

$$\Sigma_3 = \frac{1}{2} yz^2 \frac{\partial^3 \Delta}{\partial y \partial z^2} + yz \frac{\partial^2 \Delta}{\partial y \partial z}, \qquad \Sigma_4 = \frac{1}{2} zy^2 \frac{\partial^3 \Delta}{\partial z \partial y^2} + yz \frac{\partial^2 \Delta}{\partial y \partial z}. \tag{5.7}$$

In order to complete the analysis, we examine the first equation of (4.5), which couples the two scalar potentials. It yields six linear algebraic equations for the WDVV coefficients, but only three are independent. Abbreviating

$$a = (y\partial_2 - \partial_1) U, \qquad b = (z\partial_3 - \partial_1) U, \qquad m = (x^1 \partial_2 \partial_2 + \partial_1) U,$$
$$p = (x^1 \partial_3 \partial_3 + \partial_1) U, \qquad n = x^1 \partial_2 \partial_3 U, \tag{5.8}$$

one finds

$$\Sigma_1 = -\frac{m}{ay^2} - \frac{b}{a} \Sigma_4, \qquad \Sigma_2 = -\frac{p}{bz^2} + \frac{an}{b^2 yz} + \frac{a^2}{b^2} \Sigma_4, \qquad \Sigma_3 = -\frac{n}{byz} - \frac{a}{b} \Sigma_4. \tag{5.9}$$

In order to fix the last missing coefficient Σ_4, one is to analyze the WDVV equations, i.e. the second relation in (4.5). Using the explicit representation (5.6) it is straightforward to verify that among the six nontrivial WDVV equations at $n=3$ only one is independent, namely

$$W^{22p} W^{33p} = W^{23p} W^{23p}. \tag{5.10}$$

With the help of (5.9) this reduces to a linear equation, which determines Σ_4 as

$$\Sigma_4 = \frac{1}{18y} \left(\frac{9}{y-z} + \frac{6}{y+z+yz} - \frac{2}{2y-z-yz} + \frac{4}{2z-y-yz} + \frac{1}{2yz-y-z} \right), \tag{5.11}$$

and therewith Σ_1, Σ_2 and Σ_3.

The fact that for the three-body problem the WDVV equation (5.10) turns out to be linear can be understood in a different way. One can extract from the WDVV equation linear consequences which, along with other equations in (4.3)–(4.6), already contain all the information in (5.10). Indeed, let us differentiate the middle equation in (4.10),

$$(\partial_j \partial_i U)(\partial_i U) \;=\; \partial_j V_B\,, \qquad (5.12)$$

and contract the first equation in (4.10) with $\partial_i U$,

$$\partial_j V_B \;=\; W_{ijk}(\partial_i U)(\partial_k U) \;. \qquad (5.13)$$

Now contracting the WDVV equation with $(\partial_i U)(\partial_j U)$ and taking into account the first equation in (4.10) one gets the linear equations

$$(\partial_i \partial_k U)(\partial_j \partial_k U) - W_{ijk}\,\partial_k V_B \;=\; 0 \;. \qquad (5.14)$$

It is straightforward to verify that only one component in (5.14) is independent and contains just the same information as (5.10).

Having fixed the WDVV coefficients algebraically, we are now in a position to find the potential F. Substituting (5.9) and (5.11) into (5.7), one obtains for the single function Δ a system of partial differential equations of the Euler type. The standard change of variables

$$y \;=\; \mathrm{e}^t \qquad \text{and} \qquad z \;=\; \mathrm{e}^s \qquad (5.15)$$

turns it into a system of partial differential equations with constant coefficients. The latter is readily integrated by conventional means (see e.g. [46]) and yields the following free energy,

$$\begin{aligned}
F(x^1, x^2, x^3) \;=\; & -\frac{1}{6}(x^1{+}x^2{+}x^3)^2 \ln|x^1{+}x^2{+}x^3| \;+ \\
& -\frac{1}{4}\sum_{i<j}(x^i{-}x^j)^2 \ln|x^i{-}x^j| \;+\; \frac{1}{36}\sum_{\substack{i<j \\ i\neq k\neq j}}(x^i{+}x^j{-}2x^k)^2 \ln|x^i{+}x^j{-}2x^k|\,,
\end{aligned}$$
$$(5.16)$$

revealing the values

$$h_0 = \frac{1}{3}\,, \quad h_1 = \frac{1}{2}\,, \quad h_2 = -\frac{1}{18} \qquad (5.17)$$

in the ansatz (4.19) for the three types of roots in (4.27). The relative particle motion is translation invariant. Note that each sum contains three terms, so that the result is totally symmetric in $\{x^1, x^2, x^3\}$. Six constants of integration enter a polynomial quadratic in x, which can be discarded since F is defined up to such a polynomial. The quantum correction to the Calogero potential finally reads

$$O_2 \;=\; \frac{3}{8}(x^1{+}x^1{+}x^3)^{-2} \;+\; \frac{1}{4}\sum_{i<j}(x^i{-}x^j)^{-2} \;+\; \frac{1}{12}\sum_{\substack{i<j \\ i\neq k\neq j}}(x^i{+}x^j{-}2x^k)^{-2} \;. \qquad (5.18)$$

For the reader's convenience we display the corresponding WDVV coefficients in appendix B.

The $\mathcal{N}{=}4$ superconformal extension of the three-particle Calogero system produced a unique G_2-type integrable model with one free coupling and particular three-body interactions [29]. Despite the latter, we call this a Calogero model because its bosonic classical potential V_B is just the (A-type) Calogero one. This terminology differs from the one of Wyllard [34], who allowed for three-body interactions in U and V_B from the outset. Our model agrees with his second one-parameter solution.

5.2 A four-body $\mathcal{N}{=}4$ superconformal model

In this section we consider the second strategy outlined after (4.28) and construct a four-body $\mathcal{N}{=}4$ superconformal model starting from a solution F to the WDVV equations. For $n{=}4$ we make the following ansatz for the potential F,

$$
\begin{aligned}
F(x^1, x^2, x^3, x^4) \;=\; & -\frac{1}{2} h_0 \, (x^1 + x^2 + x^3 + x^4)^2 \ln|x^1 + x^2 + x^3 + x^4| \;+ \\
& -\frac{1}{2} h_1 \sum_{\substack{j>i<k<l \\ k \neq j \neq l}} (x^i + x^j - x^k - x^l)^2 \ln|x^i + x^j - x^k - x^l|
\end{aligned}
\tag{5.19}
$$

where the permutation sum has three terms. Note that the chosen positive root vectors

$$
\vec{n}^0 = (1,1,1,1), \quad \vec{n}^1 = (1,1,-1,-1), \quad (1,-1,1,-1), \quad (1,-1,-1,1)
\tag{5.20}
$$

give translation-invariant relative motion and form an orthogonal set, i.e. we look at a minimal model with a $A_1 \oplus A_1 \oplus A_1 \oplus A_1$ root system. Substituting the ansatz into (4.11), one learns that

$$
h_0 \;=\; h_1 \;=\; \frac{1}{4} \,,
\tag{5.21}
$$

in agreement with the minimal property $h_\mu^{-1} = \vec{n}^\mu \cdot \vec{n}^\mu$ from (4.22). For the case at hand one finds twenty WDVV equations, which happen to be satisfied identically for the above value of h_0 and h_1.

Let us take the corresponding ansatz for U,

$$
U \;=\; -g_0 \ln|x^1 + x^2 + x^3 + x^4| \;-\; g_1 \sum_{\substack{j>i<k<l \\ k \neq j \neq l}} \ln|x^i + x^j - x^k - x^l| \,,
\tag{5.22}
$$

where g_0 and g_1 play the role of two independent coupling constants. It is straightforward to verify that the first equation in (4.10) holds without imposing any restrictions on the form of the coupling constants. The last equation in (4.10) determines the value of the central charge as

$$
C \;=\; g_0 + 3g_1 \,,
\tag{5.23}
$$

while the second equation in (4.10) determines the form of the bosonic potential,

$$
V_B = 2g_0^2 \, (x^1 + x^2 + x^3 + x^4)^{-2} \;+\; 2g_1^2 \sum_{\substack{j>i<k<l \\ k \neq j \neq l}} (x^i + x^j - x^k - x^l)^{-2} \,,
\tag{5.24}
$$

$$
O_2 \;=\; \frac{1}{2} \, (x^1 + x^2 + x^3 + x^4)^{-2} \;+\; \frac{1}{2} \sum_{\substack{j>i<k<l \\ k \neq j \neq l}} (x^i + x^j - x^k - x^l)^{-2} \,,
\tag{5.25}
$$

in tune with the minimal expressions (4.24). Notice that g_0 and g_1 are independent and may be set to zero individually, but not their quantum corrections. This model was also found in [34].

5.3 All $\mathcal{N}{=}4$ three- and four-particle models based on $A_1 \oplus G_2$ and F_4

Let us finally make a more systematic search for $\mathcal{N}{=}4$ superconformal three- and four-particle models, where the sums in (4.19) run over particular positive root systems and all coefficients are left open. We adopt our second solution strategy and first solve the WDVV equations. The resulting admissible values for the coefficients h_μ already define all $U = 0$ models, since a vanishing U solves the first equation in (4.28) trivially. We shall encounter a free parameter t in the allowed values $h_\mu(t)$, for special values of which it is possible to turn on some g_μ in U, i.e. find nontrivial solutions to the first equation in (4.28). Motivated by the already known solutions, we allow any positive root from $A_1 \oplus G_2$ in the $n{=}3$ case and from F_4 in the $n{=}4$ case. The result of a computer analysis is given in tables 1–3.

In these tables, $\#$ is the number of positive roots obtained by permuting the entries of the displayed vector, 'type' refers to short (S) or long (L) roots, and $\times$ indicates a free coupling g_μ. The free parameter t reflects the freedom of shifting the weights between the short and the long roots.

For $n{=}3$, all models have translation-invariant relative motion, and all (except model 1 for $t{=}0$ and $t{=}\frac{1}{9}$) exploit the full G_2 root system through F. Model 1 has $U_{\rm rel} = 0$, but models 2 and 3 with a nontrivial $U_{\rm rel}$ arise at the special values of $t = -\frac{1}{18}$ and $t = \frac{1}{6}$, respectively. Model 2 was constructed in subsection 5.1, and all three indeed appear in [34].

For $n{=}4$, only models 5 and 6 feature translation-invariant relative motion, and only model 1 uses all roots of F_4. Models 1–4 have $U = 0$, and model 5 shows $U_{\rm rel} = 0$, leaving models 6, 7 and 8 with a nontrivial $U_{\rm rel}$. The latter three arise at the special point $t = 0$ of the corresponding models listed above them. Models 1–4 all intersect at $t = \frac{1}{24}$, but model 2 also agrees with model 5 at $t = 0$ (where it becomes model 6). Model 6 was presented in subsection 5.2 and also by Wyllard [34], who insisted in relative translation invariance. Furthermore, in table 4 we characterize the eight models (plus some special t values) by the subalgebra of F_4 each root system generates.

For the reader's convenience, we display the bosonic potentials for the models 5–8:

$$
\begin{aligned}
V_5 &= \frac{2\,\widetilde{g}_0^2}{(x_1+x_2+x_3+x_4)^2} \;+\; \sum_{3\ {\rm perms}} \frac{\frac{1}{2}(1-16\,t^2)^2\hbar^2}{(x_i+x_j-x_k-x_l)^2} \;+\; \sum_{6\ {\rm perms}} \frac{16\,t^2\hbar^2}{(x_i-x_j)^2} \;+\; O(\psi^2,\psi^4)\,, \\
V_6 &= \frac{2\,\widetilde{g}_0^2}{(x_1+x_2+x_3+x_4)^2} \;+\; \sum_{3\ {\rm perms}} \frac{2\,\widetilde{g}_1^2}{(x_i+x_j-x_k-x_l)^2} \;+\; O(\psi^2,\psi^4)\,, \\
V_7 &= \sum_{4\ {\rm perms}} \frac{2\,\widetilde{g}_2^2}{(x_i+x_j+x_k-x_l)^2} \;+\; O(\psi^2,\psi^4)\,, \\
V_8 &= \sum_{4\ {\rm perms}} \frac{2\,\widetilde{g}_3^2}{x_i^2} \;+\; O(\psi^2,\psi^4)\,,
\end{aligned}
$$

$$\tag{5.26}$$

$A_1 \oplus G_2$			model 1		model 2		model 3	
pos. root $\vec{n}^\mu$	#	type	g_μ	h_μ	g_μ	h_μ	g_μ	h_μ
$(1,\ 1,\ 1)$	1	$-$	$\times$	$\frac{1}{3}$	$\times$	$\frac{1}{3}$	$\times$	$\frac{1}{3}$
$(1,-1,\ 0)$	3	S	0	$\frac{1}{3}-3t$	$\times$	$\frac{1}{2}$	0	$-\frac{1}{6}$
$(1,\ 1,-2)$	3	L	0	t	0	$-\frac{1}{18}$	$\times$	$\frac{1}{6}$

Table 1: Three-particle models

F_4			model 1		model 2		model 3		model 4	
pos. root $\vec{n}^\mu$	#	type	g_μ	h_μ	g_μ	h_μ	g_μ	h_μ	g_μ	h_μ
$(1,\ 1,\ 1,\ 1)$	1	S	0	$\frac{1}{12}-2t$	0	$\frac{1}{4}-6t$	0	0	0	0
$(1,\ 1,-1,-1)$	3	S	0	$\frac{1}{12}-2t$	0	$\frac{1}{4}-6t$	0	0	0	0
$(1,\ 1,\ 1,-1)$	4	S	0	$\frac{1}{12}-2t$	0	0	0	$\frac{1}{4}-6t$	0	0
$(2,\ 0,\ 0,\ 0)$	4	S	0	$\frac{1}{12}-2t$	0	0	0	0	0	$\frac{1}{4}-6t$
$(2,\ 2,\ 0,\ 0)$	6	L	0	t	0	t	0	t	0	t
$(2,-2,\ 0,\ 0)$	6	L	0	t	0	t	0	t	0	t

Table 2: Four-particle $U{=}0$ models

F_4 continued			model 5		model 6		model 7		model 8	
pos. root $\vec{n}^\mu$	#	type	g_μ	h_μ	g_μ	h_μ	g_μ	h_μ	g_μ	h_μ
$(1,\ 1,\ 1,\ 1)$	1	S	$\times$	$\frac{1}{4}$	$\times$	$\frac{1}{4}$	0	0	0	0
$(1,\ 1,-1,-1)$	3	S	0	$\frac{1}{4}-4t$	$\times$	$\frac{1}{4}$	0	0	0	0
$(1,\ 1,\ 1,-1)$	4	S	0	0	0	0	$\times$	$\frac{1}{4}$	0	0
$(2,\ 0,\ 0,\ 0)$	4	S	0	0	0	0	0	0	$\times$	$\frac{1}{4}$
$(2,\ 2,\ 0,\ 0)$	6	L	0	0	0	0	0	0	0	0
$(2,-2,\ 0,\ 0)$	6	L	0	t	0	0	0	0	0	0

Table 3: Four-particle $U{\neq}0$ models

model number	1	$t{=}0,\frac{1}{24}$	2, 3, 4	$t{=}\frac{1}{24}$	5	$t{=}\frac{1}{16}$	6, 7, 8
# pos. roots	24	12	16	12	10	7	4
dimension	52	28	36	28	24	18	12
subalgebra	F_4	D_4	B_4	D_4	$A_1{\oplus}B_3$	$A_1{\oplus}A_3$	A_1^4

Table 4: Four-particle root systems

with $O(\psi^2,\psi^4)$ being Weyl ordered and $\widetilde{g}_\mu^2 = g_\mu^2 + \frac{1}{4}\hbar^2$. Finally, $C = \sum_\mu \#_\mu g_\mu$.

6. Conclusion

In this paper the transformation of generic conformal multi-particle mechanics into a non-interacting system with nonlocal conformal symmetry [40] was extended to accommodate $\mathcal{N}{=}4$ supersymmetry. This step facilitates the construction of new $su(1,1|2)$ invariant many-body systems. More concretely, for a potential ansatz quartic in the fermionic coordinates, the closure of the superalgebra gave rise to a set of "structure equations" (4.9) and (4.10) for two scalar (pre)potentials U and F determining the potential V, including quantum corrections.

For the n-body functions U and F we made an ansatz based on the choice of a root system, with couplings g and h, respectively, for each kind of root. This reduced the structure equations to (4.28) with (4.23), i.e. quadratic algebraic WDVV-type equations for $\partial\partial\partial F$ and linear differential equations for ∂U in the F background. We fully analyzed these equations for the case of three and four particles and found various solutions, based on the root systems of $A_1 \oplus G_2$ and F_4, respectively. The G_2-type models are identical to those of Wyllard [34], whereas in the F_4 case we extend his result (our model 6) by several other solutions featuring translationally non-invariant relative particle motion. Results based on higher-dimensional root systems will be reported elsewhere.

For three particles, the generality of our ansatz was proved by explicit integration of the structure equations (4.9) and (4.10). With a growing number of particles, this becomes rather involved because these equations are very rigid. For a general solution (unbiased by the root-system ansatz) beyond $n{=}3$ a more advanced technique is needed.

Turning to possible further developments, it would be interesting to generalize the present analysis to models exhibiting a $D(2,1|\alpha)$ symmetry and to the $\mathcal{N}{=}8$ superconformal models constructed recently in [14, 17]. One may also attempt to construct an off-shell superfield description. Finally, it is an open question whether the integrability of $\mathcal{N}{=}4$ superconformal multi-particle models is tied to the root systems of certain Lie superalgebras.

A. Fermionic Weyl ordering

Given fermionic operators $\psi_1, \ldots, \psi_n$, the Weyl-ordered product is defined as follows,

$$\langle \psi_1 \psi_2 \rangle = \frac{1}{2}\big(\psi_1 \psi_2 - \psi_2 \psi_1\big),$$

$$\langle \psi_1 \psi_2 \psi_3 \rangle = \frac{1}{3}\big(\psi_1 \langle \psi_2 \psi_3 \rangle + \psi_2 \langle \psi_3 \psi_1 \rangle + \psi_3 \langle \psi_1 \psi_2 \rangle\big),$$

$$\langle \psi_1 \psi_2 \psi_3 \psi_4 \rangle = \frac{1}{4}\big(\psi_1 \langle \psi_2 \psi_3 \psi_4 \rangle - \psi_2 \langle \psi_3 \psi_4 \psi_1 \rangle + \psi_3 \langle \psi_4 \psi_1 \psi_2 \rangle - \psi_4 \langle \psi_1 \psi_2 \psi_3 \rangle\big)$$

etc., such that for any two neighboring operators one has

$$\langle \ldots \psi_i \psi_j \ldots \rangle = -\langle \ldots \psi_j \psi_i \ldots \rangle .$$

For deriving (3.12) it is convenient to pass from Weyl-ordered operators to qp-ordered ones. In particular, for completely symmetric functions M_{ij} and L_{ijkl} one has

$$M_{ij}\,\langle \psi_\alpha^i \bar{\psi}^{j\alpha} \rangle = M_{ij}\,\psi_\alpha^i \bar{\psi}^{j\alpha} - \hbar\, M_{kk},$$

$$W_{ijk}\,\langle \psi^{i\beta} \psi_\beta^j \bar{\psi}_\alpha^k \rangle = W_{ijk}\,\psi^{i\beta} \psi_\beta^j \bar{\psi}_\alpha^k - \hbar\, W_{kki}\,\psi_\alpha^i,$$

$$W_{ijk}\,\langle \psi^{i\alpha} \bar{\psi}_\beta^j \bar{\psi}^{k\beta} \rangle = W_{ijk}\,\psi^{i\alpha} \bar{\psi}_\beta^j \bar{\psi}^{k\beta} + \hbar\, W_{kki}\,\bar{\psi}^{i\alpha},$$

$$L_{ijkl}\,\langle \psi^{i\alpha} \psi_\alpha^j \bar{\psi}_\beta^k \bar{\psi}^{l\beta} \rangle = L_{ijkl}\,\psi^{i\alpha} \psi_\alpha^j \bar{\psi}_\beta^k \bar{\psi}^{l\beta} - 2\hbar\, L_{ijkk}\,\psi_\alpha^i \bar{\psi}^{j\alpha} + \hbar^2\, L_{kkpp}.$$

B. WDVV coefficients

Here we present the explicit form of the WDVV coefficients for the three-body $\mathcal{N}=4$ superconformal Calogero model (5.16):

$$18 W_{112} = +\frac{9}{x_1-x_2} - \frac{4}{2x_1-x_2-x_3} + \frac{2}{2x_2-x_1-x_3} - \frac{1}{2x_3-x_1-x_2} - \frac{6}{x_1+x_2+x_3},$$

$$18 W_{113} = +\frac{9}{x_1-x_3} - \frac{4}{2x_1-x_2-x_3} - \frac{1}{2x_2-x_1-x_3} + \frac{2}{2x_3-x_1-x_2} - \frac{6}{x_1+x_2+x_3},$$

$$18 W_{122} = -\frac{9}{x_1-x_2} + \frac{2}{2x_1-x_2-x_3} - \frac{4}{2x_2-x_1-x_3} - \frac{1}{2x_3-x_1-x_2} - \frac{6}{x_1+x_2+x_3},$$

$$18 W_{123} = +\frac{2}{2x_1-x_2-x_3} + \frac{2}{2x_2-x_1-x_3} + \frac{2}{2x_3-x_1-x_2} - \frac{6}{x_1+x_2+x_3},$$

$$18 W_{133} = -\frac{9}{x_1-x_3} + \frac{2}{2x_1-x_2-x_3} - \frac{1}{2x_2-x_1-x_3} - \frac{4}{2x_3-x_1-x_2} - \frac{6}{x_1+x_2+x_3},$$

$$18 W_{223} = +\frac{9}{x_2-x_3} - \frac{1}{2x_1-x_2-x_3} - \frac{4}{2x_2-x_1-x_3} + \frac{2}{2x_3-x_1-x_2} - \frac{6}{x_1+x_2+x_3},$$

$$18 W_{233} = -\frac{9}{x_2-x_3} - \frac{1}{2x_1-x_2-x_3} + \frac{2}{2x_2-x_1-x_3} - \frac{4}{2x_3-x_1-x_2} - \frac{6}{x_1+x_2+x_3},$$

$$18 W_{111} = -\frac{9}{x_1-x_2} - \frac{9}{x_1-x_3} + \frac{8}{2x_1-x_2-x_3} - \frac{1}{2x_2-x_1-x_3} - \frac{1}{2x_3-x_1-x_2} - \frac{6}{x_1+x_2+x_3},$$

$$18 W_{222} = +\frac{9}{x_1-x_2} - \frac{9}{x_2-x_3} - \frac{1}{2x_1-x_2-x_3} + \frac{8}{2x_2-x_1-x_3} - \frac{1}{2x_3-x_1-x_2} - \frac{6}{x_1+x_2+x_3},$$

$$18 W_{333} = +\frac{9}{x_1-x_3} + \frac{9}{x_2-x_3} - \frac{1}{2x_1-x_2-x_3} - \frac{1}{2x_2-x_1-x_3} + \frac{8}{2x_3-x_1-x_2} - \frac{6}{x_1+x_2+x_3}.$$

The quantum correction $\hbar^2 O_2 = \frac{\hbar^2}{8} W_{ijk} W_{ijk}$ to the two-body Calogero potential was given in (5.18) and involves three-body interactions.

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